Tutorial Note for Math2012E

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1 Curves in space

To determine a point in three-dimensional space, we need three coordinates. Similarly to determine a curve in in three-dimensional space, we need three functions. And this leads to the concept: vector-valued functions.

- equation:
$$\vec{r}(t) = (x(t), y(t), z(t))$$

- limit:
$$\begin{aligned}
&\lim_{t\to t_0} \vec{r}(t) = \vec{L} \\
\Leftrightarrow &\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |\vec{r}(t) - \vec{L}| < \epsilon \text{ whenever } 0 < |t - t_0| < \delta \\
\Leftrightarrow &\lim_{t\to t_0} x(t) = L_1, \lim_{t\to t_0} y(t) = L_2, \lim_{t\to t_0} z(t) = L_3
\end{aligned}$$

- The above definition can be extended to any metric space

- derivative:
$$\frac{d\vec{r}(t)}{dt} = (x'(t), y'(t), z'(t))$$
i.e. differential at each component.

- some names
  - velocity is the dirivative of position: $\vec{v} = \frac{d\vec{r}}{dt}$
  - speed is the magnitude of velocity: speed= $|\vec{v}|$
  - acceleration is the derivative of velocity: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

- differential rules
  - linear to addition
  - scalar multiplication
$$\frac{d}{dt}[f(t)\vec{r}(t)] = f'(t)\vec{r}(t) + f(t)\frac{d\vec{r}(t)}{dt}$$

- chain rule

$$\frac{d}{dt}[\vec{r}(f(t))] = f'(t)\frac{d\vec{r}(t)}{dt}$$

- dot/cross product

$$\frac{d}{dt}[\vec{u}(t) * \vec{v}(t)] = \frac{d\vec{u}(t)}{dt} * \vec{v}(t) + \vec{u}(t) * \frac{d\vec{v}(t)}{dt}$$

Here $*$ denotes dot/cross product

- vector functions of constant length: $|\vec{r}| = c \Leftrightarrow \vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

2 integrals of vector functions, projctile motion

- Integral: Just integrate by components.
- ideal projectile motion equation:

$$\vec{r} = ((v_0 \cos\alpha)t, (v_0 \sin\alpha)t - \frac{1}{2}gt^2)$$

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  - Maximum height: $y_{max} = \frac{(v_0 \sin\alpha)^2}{2g}$
  - Flight time: $t = \frac{2v_0 \sin\alpha}{g}$
  - Range: $R = \frac{v_0^2}{g}\sin 2\alpha$

3 Arc Length in space

- $ds^2 = \left[x'(t)^2 + y'(t)^2 + z'(t)^2\right] dt^2$
- $L = \int_a^b ds$

4 Curvature and normal vector of a curve

- unit tangent vector

$$\vec{T} = \frac{d\vec{r}}{ds}$$

- curvature function

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

- Formula for calculating curvature

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^2}$$

- At point where $\kappa \neq 0$, the principap unit normal vector for a smooth curve in the plane is
$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$
- Formula for calculating N
$$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{|\frac{d\vec{T}}{dt}|}$$
- circle of curvature for plane curves
  - tangent to the curve at P
  - has the same curvature the curve has at P
  - lie toward the concave or inner side of the curve
  - Radius of curvature $= \rho = \frac{1}{\kappa}$

5 Tangential and Normal components of acceleration

- TNB frame/Frenet frame
$$\vec{T}, \vec{N}, \vec{B} = \vec{T} \times \vec{N}$$
- write acceleration vector as following :
$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$
  - tangential scalar component
$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\vec{v}|$$
  - normal scalar component
$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\vec{v}|^2$$
- Torsion
$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$
- computation foumulas
$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

6 planet move

Kapler's Law

- first(ellipse law)
$$e = \frac{r_0 v_0^2}{GM} - 1$$
- second(Equal Area Law): radius vector from the sun to a planet sweeps out equal areas in equal times.
- Third(Time-Distance law)
$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$